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THERMOELASTIC WAVES IN A PERIODICALLY LAMINATED MEDIUM†

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Abstract—The oblique propagation of time–harmonic waves in a periodically laminated composite is studied. Governing equations are those of the coupled thermoelasticity theory for plane strain. The exact dispersion relation in the form of a determinant of order twelve is presented, and several numerical cases are included. Results indicating differing amounts of dispersion and attenuation for various angles of propagation are compared with results from an isothermal analysis.

INTRODUCTION

WAVE propagation in periodic laminated media is of practical importance in the analysis and design of composite materials for severe environments. The purpose of this investigation is to consider one aspect of the total problem, i.e. to determine what effect the coupling of the thermal and elastic fields has on the response of a laminated composite. This is accomplished by a study of the propagation of harmonic waves through a medium that obeys the equations of coupled thermoelasticity, e.g. Biot [1] and Chadwick [2]. A formulation due to Deresiewicz [3] provides the governing equations in terms of potential functions, and substitution of a traveling wave yields the dispersive properties of the medium. Also resulting from this analysis is a measure of the attenuation as the disturbance propagates through the composite.

When the thermoelastic coupling can be neglected, the results reduce to those reported by previous investigators. Rytov [4] published the dispersion relations for harmonic waves propagating along or across the laminations and also included the relation for waves propagating obliquely in a composite with thin laminations. Sun *et al.* [5, 6] developed an approximate theory for a laminated medium and included comparisons with the exact results for waves propagating along and across the laminations. Sve [7] showed that this approximate theory also provides phase velocities that substantially agree with those from the exact theory for waves traveling obliquely in the composite. The approximate theory has also been used [8–10] to solve several transient problems involving one-dimensional pulse propagation in laminated composites. Recently, Grot and Achenbach [11] have extended the approximate theory to include thermoelastic coupling and viscoelastic effects.

ANALYSIS

Basic equations

The equations of coupled thermoelasticity have been reduced by Deresiewicz [3] to three differential equations after applying the Helmholtz theorem. In the case of plane strain, the

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displacements become

$$u_1 = \phi_{1,1}+\phi_{2,1}+\phi_{3,2}, \qquad u_2 = \phi_{1,2}+\phi_{2,2}-\phi_{3,1}, \tag{1}$$

while the stresses for isotropic material are

$$\sigma_{ij} = \lambda S_{ij}(u_{1,1}+u_{2,2})+\mu(u_{i,j}+u_{j,i})-S_{ij}(3\lambda+2\mu)\alpha_t T \tag{2}$$

In these equations, u_1 and u_2 are the displacements in the x_1 and x_2 directions, respectively; ϕ_1 and ϕ_2 are scalar potential functions, while ϕ_3 is the vector potential function; and Lamé's constants have been denoted by λ and μ. The coefficient of thermal expansion is α_t and the change in temperature is designated T. The subscript notation has been used with subscripts following a comma indicating partial differentiation, and S_{ij} represents the Kronecker delta. For a harmonic time variation

$$\phi_j = \bar{\Phi}_j\, e^{-i\omega t}, \qquad j = 1,2,3, \tag{3}$$

with frequency ω, the governing set of equations becomes

$$(\nabla^2+\delta_j^2)\bar{\Phi}_j = 0, \qquad j = 1,2 \tag{4}$$

$$(\nabla^2+\omega^2\rho/\mu)\bar{\Phi}_3 = 0 \tag{5}$$

where

$$\delta_j^2 = \frac{\omega^2\rho}{2(\lambda+2\mu)}\left\{1-\frac{1+\varepsilon}{\Lambda}\pm\left[1+\frac{2(1-\varepsilon)}{\Lambda}+\left(\frac{1+\varepsilon}{\Lambda}\right)^2\right]^{\frac{1}{2}}\right\} \tag{6}$$

and

$$\Lambda = ia^2\omega\rho/(\lambda+2\mu). \tag{7}$$

Here, the mass density is ρ, the thermal diffusivity is a^2 and the coupling parameter is

$$\varepsilon = \frac{\alpha_t^2(3\lambda+2\mu)^2 T_0}{c_v(\lambda+2\mu)} \tag{8}$$

where the specific heat at constant volume per unit volume is c_v. The temperature deviation from the equilibrium temperature T_0 is

$$T = \frac{1-\nu}{\alpha_t(1+\nu)}\sum_{j=1}^{2}\left(\frac{\omega^2\rho}{\lambda+2\mu}-\delta_j^2\right)\bar{\Phi}_j \tag{9}$$

with ν denoting Poisson's ratio.

Solution for laminated medium

In this case, the solution may be represented by

$$(\bar{\Phi}_{1j}, \bar{\Phi}_{2j}, \bar{\Phi}_{3j}) = (f_j, g_j, h_j)\exp[ik(n_1x_1+n_2x_2)] \tag{10}$$

with the layer index $j = 1$ or 2. The direction of propagation for the laminated medium shown in Fig. 1 is determined by the direction cosines $n_1 = \cos\alpha$ and $n_2 = \sin\alpha$, while the wave number is designated by k. Substitution of equation (10) into equations (4) and (5)

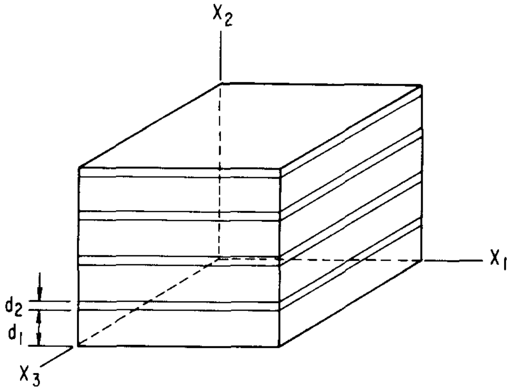

FIG. 1. Periodically laminated medium.

leads to ordinary differential equations for the unknown functions f_j, g_j and h_j. The solutions are

$$f_j(x_2) = A_j \exp[-ikx_2(n_2+\alpha_j)] + B_j \exp[-ikx_2(n_2-\alpha_j)], \tag{11}$$

$$g_j(x_2) = E_j \exp[-ikx_2(n_2+\gamma_j)] + F_j \exp[-ikx_2(n_2-\gamma_j)], \tag{12}$$

$$h_j(x_2) = C_j \exp[-ikx_2(n_2+\beta_j)] + D_j \exp[-ikx_2(n_2-\beta_j)], \tag{13}$$

where

$$\alpha_j = [(\bar{\delta}_{1j}^2/k^2) - n_1^2]^{\frac{1}{2}} = [(\delta_{1j}^2/\xi^2) - n_1^2]^{\frac{1}{2}}, \tag{14}$$

$$\gamma_j = [\bar{\delta}_{2j}^2/k^2) - n_1^2]^{\frac{1}{2}} = [(\delta_{2j}^2/\xi^2) - n_1^2]^{\frac{1}{2}}, \tag{15}$$

$$\beta_1 = [(\rho_1\omega^2/\mu_1 k^2) - n_1^2]^{\frac{1}{2}} = [(\Omega^2/\xi^2) - n_1^2]^{\frac{1}{2}}, \tag{16}$$

$$\beta_2 = [(\rho_2\omega^2/\mu_2 k^2) - n_1^2]^{\frac{1}{2}} = [(\theta\Omega^2/\gamma\xi^2) - n_1^2]^{\frac{1}{2}}. \tag{17}$$

Nondimensional quantities have been introduced as

$$\theta = \rho_2/\rho_1, \qquad \gamma = \mu_2/\mu_1, \qquad \xi = kd_2, \qquad \Omega = \omega d_2/(\mu_1/\rho_1)^{\frac{1}{2}},$$

$$\delta_{ij}^2 = \bar{\delta}_{ij}^2 d_2^2 = \frac{\omega^2 \rho_j d_2^2}{2(\lambda_j + 2\mu_j)} \left\{ 1 - \frac{1+\varepsilon_j}{\Lambda_j} \pm \left[1 + \frac{2(1-\varepsilon_j)}{\Lambda_j} + \left(\frac{1+\varepsilon_j}{\Lambda_j} \right)^2 \right]^{\frac{1}{2}} \right\}. \tag{18}$$

The notation ε_j refers to the coupling parameter defined by equation (8) for each layer.

Boundary and periodicity conditions

Continuity of displacements, stresses, temperature and heat flux at an interface must be insured. This is accomplished by the matching conditions

$$u_1(0^+) = u_1(0^-), \qquad u_2(0^+) = u_2(0^-), \qquad \sigma_{12}(0^+) = \sigma_{12}(0^-),$$

$$\sigma_{22}(0^+) = \sigma_{22}(0^-), \qquad T(0^+) = T(0^-), \qquad K_1 \frac{\partial T}{\partial x_2}(0^+) = K_2 \frac{\partial T}{\partial x_2}(0^-), \tag{19}$$

where 0^+ and 0^- are values of x_2 near zero and K represents the thermal conductivity.

Six additional conditions are available from periodicity considerations. The amplitudes of the quantities appearing in equation (19) are required to be periodic since f_j, g_j, h_j are periodic by virtue of Bloch's theorem. The problem then degenerates to two adjacent layers with twelve conditions, leading to a nontrivial solution.

Frequency equation

Substitution into these twelve conditions yields a set of homogeneous equations. For a nontrivial solution, the determinant of the coefficients is set to zero and yields

$$\begin{vmatrix}
-n_1 & -n_1 & \beta_1 & -\beta_1 & -n_1 & -n_1 & n_1 & n_1 & -\beta_2 & \beta_2 & n_1 & n_1 \\
-\alpha_1 & \alpha_1 & -n_1 & -n_1 & -\gamma_1 & \gamma_1 & \alpha_2 & -\alpha_2 & n_1 & n_1 & \gamma_2 & -\gamma_2 \\
n_1^2-\beta_1^2 & n_1^2-\beta_1^2 & -2n_1\beta_1 & 2n_1\beta_1 & n_1^2-\beta_1^2 & n_1^2-\beta_1^2 & (\beta_2^2-n_1^2)\gamma & (\beta_2^2-n_1^2)\gamma & 2n_1\beta_2\gamma & -2n_1\beta_2\gamma & (\beta_2^2-n_1^2)\gamma & (\beta_2^2-n_1^2)\gamma \\
2n_1\alpha_1 & -2n_1\alpha_1 & n_1^2-\beta_1^2 & n_1^2-\beta_1^2 & 2n_1\gamma_1 & -2n_1\gamma_1 & -2n_1\alpha_2\gamma & 2n_1\alpha_2\gamma & (\beta_2^2-n_1^2)\gamma & (\beta_2^2-n_1^2)\gamma & -2n_1\gamma_2\gamma & 2n_1\gamma_2\gamma \\
\eta_1^2 & \eta_1^2 & 0 & 0 & \eta_2^2 & \eta_2^2 & -\eta_3^2\Delta & -\eta_3^2\Delta & 0 & 0 & -\eta_4^2\Delta & -\eta_4^2\Delta \\
\alpha_1\eta_1^2 & -\alpha_1\eta_1^2 & 0 & 0 & \gamma_1\eta_2^2 & -\gamma_1\eta_2^2 & -\alpha_2\eta_3^2\Delta\sigma & \alpha_2\eta_3^2\Delta\sigma & 0 & 0 & -\gamma_2\eta_4^2\Delta\sigma & \gamma_2\eta_4^2\Delta\sigma \\
-n_1E_1\bar{e}_1 & -n_1E_1e_1 & \beta_1E_1\bar{e}_2 & -\beta_1E_1e_2 & -n_1E_1\bar{e}_5 & -n_1E_1e_5 & n_1E_2e_3 & n_1E_2\bar{e}_3 & -\beta_2E_2e_4 & \beta_2E_2\bar{e}_4 & n_1E_2e_6 & n_1E_2\bar{e}_6 \\
-\alpha_1E_1\bar{e}_1 & \alpha_1E_1e_1 & -n_1E_1\bar{e}_2 & -n_1E_1e_2 & -\gamma_1E_1\bar{e}_5 & \gamma_1E_1e_5 & \alpha_2E_2e_3 & -\alpha_2E_2\bar{e}_3 & n_1E_2e_4 & n_1E_2\bar{e}_4 & \gamma_2E_2e_6 & -\gamma_2E_2\bar{e}_6 \\
(n_1^2-\beta_1^2)E_1\bar{e}_1 & (n_1^2-\beta_1^2)E_1e_1 & -2n_1\beta_1E_1\bar{e}_2 & 2n_1\beta_1E_1e_2 & (n_1^2-\beta_1^2)E_1\bar{e}_5 & (n_1^2-\beta_1^2)E_1e_5 & (\beta_2^2-n_1^2)\gamma E_2e_3 & (\beta_2^2-n_1^2)\gamma E_2\bar{e}_3 & 2n_1\beta_2\gamma E_2e_4 & -2n_1\beta_2\gamma E_2\bar{e}_4 & (\beta_2^2-n_1^2)\gamma E_2e_6 & (\beta_2^2-n_1^2)\gamma E_2\bar{e}_6 \\
2n_1\alpha_1E_1\bar{e}_1 & -2n_1\alpha_1E_1e_1 & (n_1^2-\beta_1^2)E_1\bar{e}_2 & (n_1^2-\beta_1^2)E_1e_2 & 2n_1\gamma_1E_1\bar{e}_5 & -2n_1\gamma_1E_1e_5 & -2n_1\alpha_2\gamma E_2e_3 & 2n_1\alpha_2\gamma E_2\bar{e}_3 & (\beta_2^2-n_1^2)\gamma E_2e_4 & (\beta_2^2-n_1^2)\gamma E_2\bar{e}_4 & -2n_1\gamma_2\gamma E_2e_6 & 2n_1\gamma_2\gamma E_2\bar{e}_6 \\
\eta_1^2E_1\bar{e}_1 & \eta_1^2E_1e_1 & 0 & 0 & \eta_2^2E_1\bar{e}_5 & \eta_2^2E_1e_5 & -\eta_3^2\Delta E_2e_3 & -\eta_3^2\Delta E_2\bar{e}_3 & 0 & 0 & -\eta_4^2\Delta E_2e_6 & -\eta_4^2\Delta E_2\bar{e}_6 \\
\alpha_1\eta_1^2E_1\bar{e}_1 & -\alpha_1\eta_1^2E_1e_1 & 0 & 0 & \gamma_1\eta_2^2E_1\bar{e}_5 & -\gamma_1\eta_2^2E_1e_5 & -\alpha_2\eta_3^2\Delta\sigma E_2e_3 & \alpha_2\eta_3^2\Lambda rE_2\bar{e}_3 & 0 & 0 & -\gamma_2\eta_4^2\Delta\sigma E_2e_6 & \gamma_2\eta_4^2\Delta\sigma E_2\bar{e}_6
\end{vmatrix} = 0. \tag{20}$$

In equation (20), the following definitions for exponentials have been introduced:

$$e_1 = \exp(i\alpha_1\xi/\zeta), \qquad e_2 = \exp(i\beta_1\xi/\zeta), \tag{21}$$

$$e_3 = \exp(i\alpha_2\xi), \qquad e_4 = \exp(i\beta_2\xi), \tag{22}$$

$$e_5 = \exp(i\xi\gamma_1/\zeta), \qquad e_6 = \exp(i\gamma_2\xi), \tag{23}$$

$$E_1 = \exp(-in_2\xi/\zeta), \qquad E_2 = \exp(in_2\xi). \tag{24}$$

The other nondimensional quantities appearing in equation (20) are

$$\sigma = K_2/K_1, \qquad \Delta = \frac{\alpha_{t1}}{\alpha_{t2}}\left(\frac{1+\nu_1}{1-\nu_1}\right)\left(\frac{1-\nu_2}{1+\nu_2}\right), \tag{25}$$

$$\eta_1^2 = \Omega^2/\delta_1 - \delta_{11}^2, \qquad \eta_2^2 = \Omega^2/\delta_1 - \delta_{21}^2, \tag{26}$$

$$\eta_3^2 = \Omega^2\theta/\gamma\delta_2 - \delta_{12}^2, \qquad \eta_4^2 = \Omega^2\theta/\gamma\delta_2 - \delta_{22}^2, \tag{27}$$

where

$$\delta_1 = 2(1-\nu_1)/(1-2\nu_1), \qquad \delta_2 = 2(1-\nu_2)/(1-2\nu_2). \tag{28}$$

Also, a thickness ratio $\zeta = d_2/d_1$ has been used, and the notation $\bar{e}_k(x)$ means $e_k(-x)$.

Since no approximations have been made, other than those inherent in the coupled theory of thermoelasticity, equation (20) represents the exact relationship between frequency and wave number for the propagation of harmonic waves in a periodic laminated medium. One can study a number of separate problems with equation (20) by assigning special values to the propagation angle or to the coupling coefficients or by limiting the frequency range.

NUMERICAL RESULTS

The determinant relates the frequency and the wave number for possible propagation modes; if the frequency assumes real positive values, the wave number is usually complex. This means that the disturbance is usually attenuating as it moves through the composite because of the conversion of a portion of the mechanical energy into heat. However, a case that produces real wave numbers for real frequencies is $\alpha = \pi/2$. The determinant degenerates into the product of two smaller determinants. An eighth-order one governs the longitudinal motion, and a fourth-order determinant consisting of the first, fourth, seventh and tenth rows and the third, fourth, ninth and tenth columns is for the transverse motion, which is not coupled with the thermal field and therefore suffers no attenuation. This case is considered in more detail later.

In order to obtain a root ξ of equation (20) for a given value of the frequency Ω, we used curve-fitting techniques based upon polynomial approximations, with initial estimates of roots available from an isothermal analysis [7]. Once a complex root was determined, an extrapolation process was used to provide an estimate for the next root. Since subroutines are available for complex determinant evaluation and complex root location, the expansion of the determinant was unnecessary. Actually, the reduction to the special cases for $\alpha = 0$, $\pi/2$ or $\varepsilon_1 = \varepsilon_2 = 0$ are easier to interpret in determinant form. It is, of course, sometimes appropriate to expand a frequency determinant if simplifications are obtained or if certain operations require such an expansion. For the purposes here, the determinant need not be expanded.

When equation (18) is written in terms of the nondimensional quantities, the expressions for layer 1 become

$$\delta_{j1}^2 = \frac{\Omega^2}{2\delta_1}\left\{1 - \frac{1+\varepsilon_1}{i\Omega\kappa_1} \pm \left[1 + \frac{2(1-\varepsilon_1)}{i\Omega\kappa_1} - \left(\frac{1+\varepsilon_1}{\Omega\kappa_1}\right)^2\right]^{\frac{1}{2}}\right\} \tag{29}$$

and for layer 2

$$\delta_{j2}^2 = \frac{\Omega^2\theta}{2\gamma\delta_2}\left\{1 - \frac{1+\varepsilon_2}{i\Omega\kappa_2} \pm \left[1 + \frac{2(1-\varepsilon_2)}{i\Omega\kappa_2} - \left(\frac{1+\varepsilon_2}{\Omega\kappa_2}\right)^2\right]^{\frac{1}{2}}\right\} \tag{30}$$

where

$$\kappa_1 = \frac{a_1^2\rho_1}{\mu_1\delta_1}\frac{(\mu_1/\rho_1)^{\frac{1}{2}}}{d_2}, \qquad \kappa_2 = \frac{a_2^2\rho_2}{\mu_2\delta_2}\frac{(\mu_1/\rho_1)^{\frac{1}{2}}}{d_2}. \tag{31}$$

The characteristic times defined in equation (31) in nondimensional form were discussed in detail by Chadwick and Sneddon [12], and representative values were reported for metals. Typical dimensionless values for these quantities as defined here are 10^{-3}–10^{-5} and smaller. This mismatch between mechanical and thermal parameters has been used by Deresiewicz [13] to provide expansions of the expressions in equations (29) and (30) in terms of a reduced frequency $\Omega\kappa_i$. Since in this case the roots were found numerically, the expressions were not expanded, and the results are valid for all frequency ranges.

A numerical example is included here to indicate the type of results to be expected for various values of the propagation angle. Attention is confined to the perturbed roots of the isothermal analysis although other roots exist, such as quasi-thermal roots. Table 1 lists the pertinent data with ε_2 being calculated from

$$\varepsilon_2 = \frac{\varepsilon_1\kappa_2}{\theta\sigma\kappa_1}\left(\frac{\delta_2\gamma}{\Delta\delta_1}\right)^2. \tag{32}$$

Table 1. Data for example

$\alpha = 0, 15, 90°$	$\sigma = 10$
$\gamma = 50$	$\kappa_1 = 0{\cdot}005$
$\theta = 3$	$\kappa_2 = 0{\cdot}002$
$\nu_1 = 0{\cdot}35$	$\alpha_{t1}/\alpha_{t2} = 1{\cdot}5$
$\nu_2 = 0{\cdot}3$	$\varepsilon_1 = 0{\cdot}4$
$\zeta = 4$	$\varepsilon_2 = 0{\cdot}03091$

Dispersion and attenuation curves for $\alpha = \pi/2$ are shown in Fig. 2, with the frequency Ω being plotted vs. the real and imaginary parts of the wave number ξ. Note that the transverse motion is not affected by the value of the thermal coupling parameters $\varepsilon_1, \varepsilon_2$. However, even though all branches remain periodic in Re(ξ), the longitudinal motion couples with the thermal field to produce attenuation and modified phase velocities if the definition for phase velocity is taken to be Ω/Re(ξ). Strictly speaking, phase velocity is a concept which stems from stationary phase analysis involving the real Ω–ξ plane, however, the term is used here even though the wavenumber is complex. The imaginary part of the wave number is negative for a branch of the dispersion curve that has a negative group velocity $d\Omega/d\xi$, while the reverse is true for a branch with positive group velocity. The group velocity referred to is for the nonabsorbing medium since group velocity loses its meaning when dissipation is included. In a transient problem, utilizing this spectra, a radiation condition would determine the proper branch. Note that the attenuation is quite large in a stop band. This is characteristic of periodic media (Brillouin [14]).

The other extreme case of propagation angle is $\alpha = 0$. Figure 3 shows the dispersion and attenuation curves for the two lowest modes. For this case, the determinant may be simplified since the motion splits into symmetric (mode 2) and antisymmetric (mode 1) portions about the midplanes of the layers. The greatest attenuation shown occurs when the real part of the wave number corresponding to the longitudinal mode reaches and

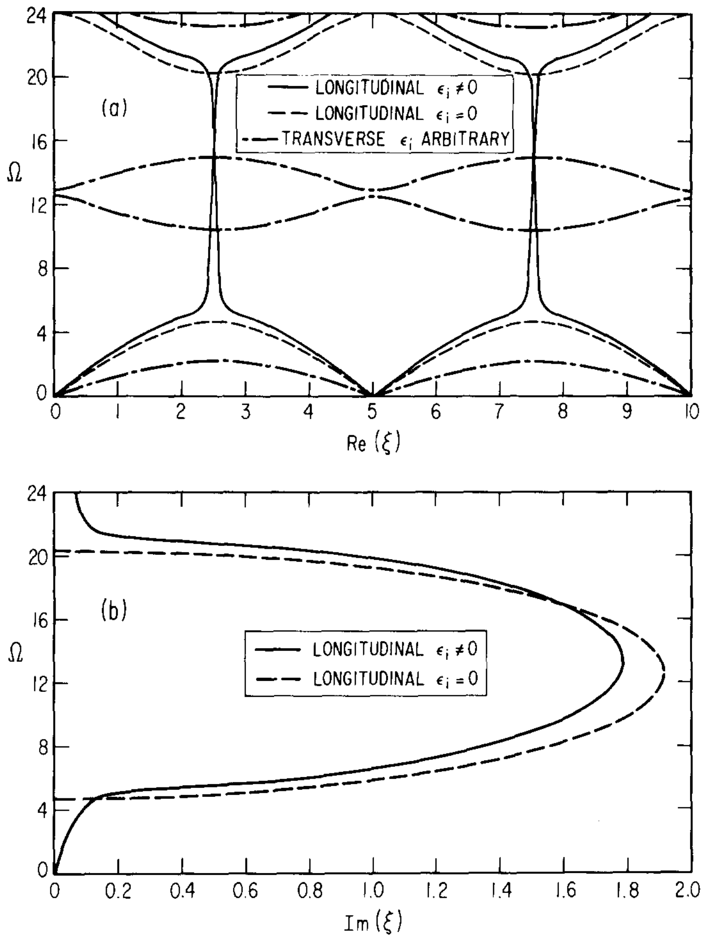

FIG. 2. Dispersion and attenuation results for $\alpha = \pi/2$.

exceeds $\pi\eta$ or equivalently when $k = \pi/(d_1 + d_2)$. Numerically, this occurs at $\text{Re}(\xi) \simeq 2{\cdot}5$, which corresponds to $\Omega \simeq 13$ for this example. This would seem to indicate that stopband behavior exists for $\alpha = 0$ when thermoelastic coupling is taken into account.

One oblique propagation case is presented in Fig. 4. Here, $\alpha = \pi/12$ was chosen to illustrate similarities and differences when compared with the $\alpha = 0$ case. Since the determinant does not degenerate for this case, the branches of the dispersion curve do not intersect even for zero coupling parameters, and the terrace-like structure prevails. The attenuation curves indicate that the modes actually switch in character as the terracing takes place and that the attenuation of the quasi-longitudinal mode increases noticeably above the terrace point.

The ratio of the thermoelastic phase velocity to the uncoupled elastic phase velocity c_T/c_E was also calculated for the three angles of propagation and is shown in Fig. 5. The thermal effect is small for low frequencies and becomes more important at higher frequencies, particularly for the predominantly longitudinal modes.

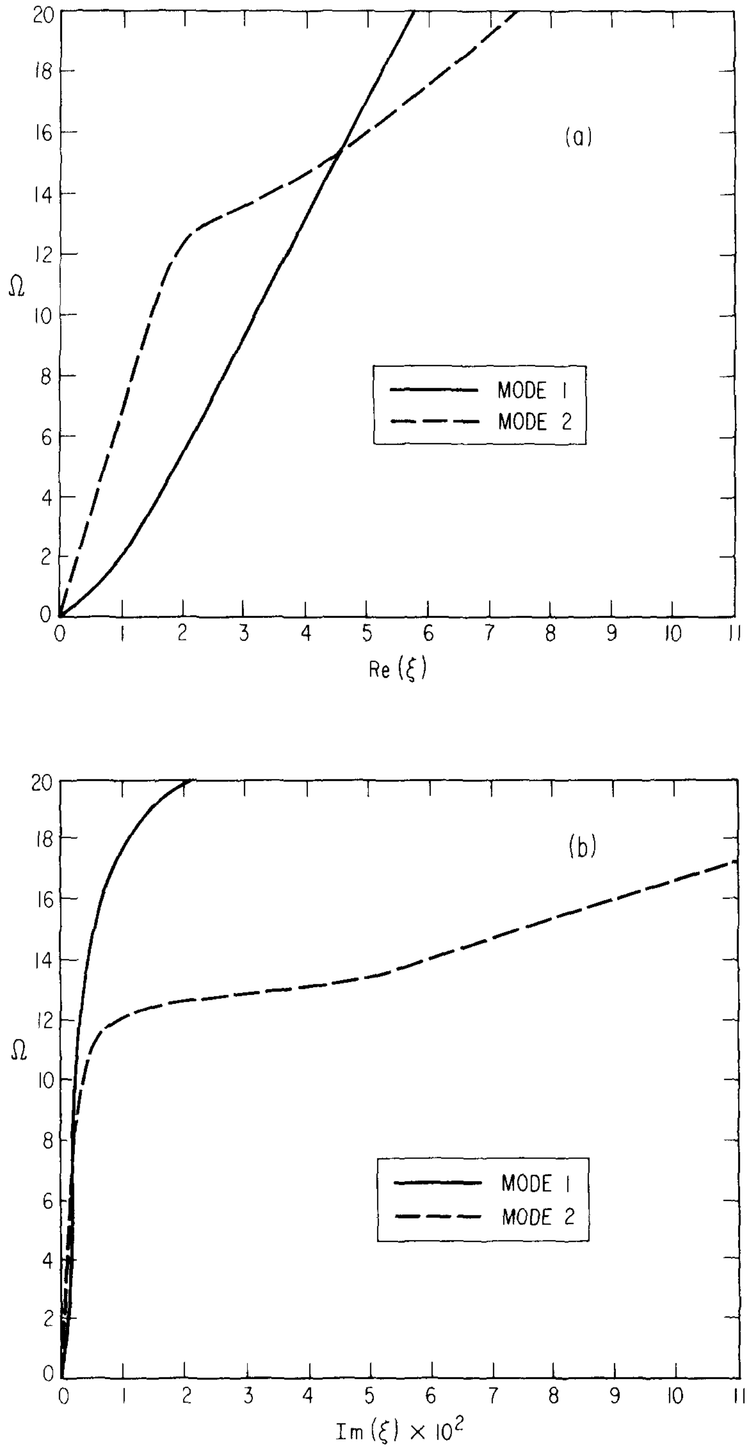

FIG. 3. Dispersion and attenuation results for $\alpha = 0$.

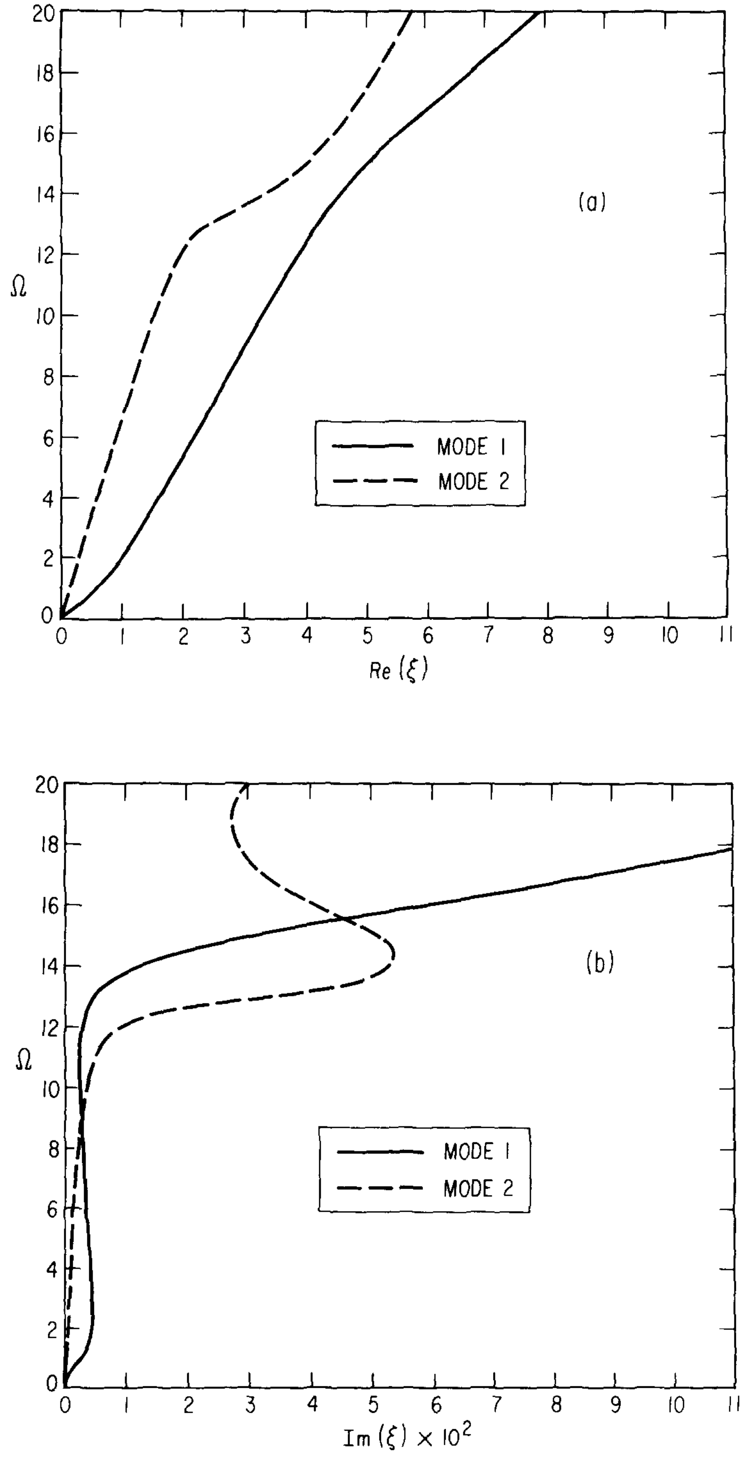

FIG. 4. Dispersion and attenuation results for $\alpha = \pi/12$.

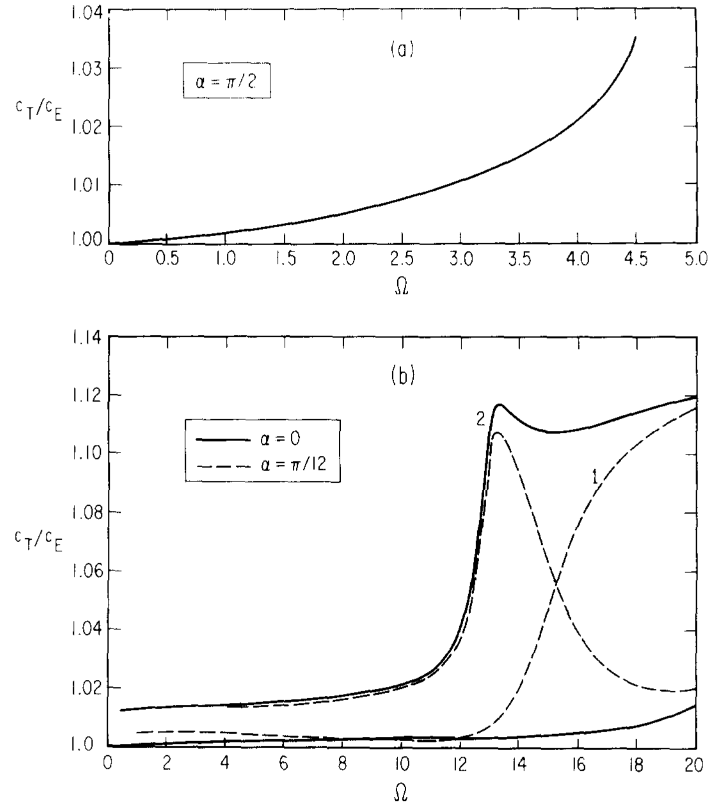

FIG. 5. Phase velocity ratio for various propagation angles.

CONCLUSIONS

The dispersion and attenuation of thermoelastic waves in a laminated composite has been determined on the basis of the coupled theory of thermoelasticity. The analytical effort provides a twelfth-order determinant relating the frequency and complex wave number that reduces properly to the isothermal case. Numerical results indicate that thermoelastic attenuation is confined primarily to the quasi-longitudinal modes. When $\alpha = \pi/2$, the modes degenerate into an attenuating longitudinal set and a nondecaying transverse set. The phase velocities and mode shapes are also influenced by the attenuation parameters, especially for large frequencies.

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Абстракт—Исследуется косое распространение гармонических во времени, волн в периодлчески слоистых материалах. Определяющие уровнения, аналогичные такимже для сопряжённой теории термоупругости, для случая плоской деформации. Даётся точная завосимость дисперсии, в форме детерминанта двенадцатого порядка и приводятся некоторые численные слчаи. Сравниваются результаты, указывающие разное количество дисперсии и затухания, для разных узлов распространения, с результатми изотермического анализа.